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On a class of stochastic reaction–diffusion equations in two space dimensions

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Abstract. We prove, for any finite time, the existence of a weak dynamics for a class of reaction–diffusion systems in two space dimensions perturbed by a white noise. This class includes models which may exhibit complex behaviour in space and time. Our approach, based on the methods of constructive quantum field theory, extends previous results established for gradient-type systems.

1. Introduction

Motivated by the approach to quantum field theory known as stochastic quantization it was proved recently [1, 2] that certain gradient-type stochastic parabolic PDEs in two space dimensions possess weak solutions in spite of infinite renormalization terms which appear in the formal expression of the equations. This result was obtained using techniques borrowed from constructive field theory. Its interest, however, goes beyond the initial context. In fact these equations represent a particular case of infinite-dimensional stochastically perturbed dynamical systems which appear naturally in several areas of science. The restriction to gradient-type equations, however, excludes those very interesting situations where the corresponding determinstic system may exhibit chaotic behaviour.

In this paper it is shown that the theory can be extended to a class of non-gradient systems which we believe includes a large number of physically relevant examples.

To characterize the equations that we are able to treat, we start from the general form†

$$\partial_t \boldsymbol{\phi}_t = \nu \Delta \boldsymbol{\phi}_t + \tilde{\boldsymbol{F}}(\phi_t) + \varepsilon \partial_t \boldsymbol{W}_t \tag{1}$$

with

$$E(\partial_t W_{t,i}(x) \partial_{t'} W_{t',k}(x')) = \delta(x-x')\delta(t-t')\delta_{ik} \qquad x, x' \in R^2 \tag{2}$$

where $\{\phi_{t,i}(x)\}_{\{i=1,\dots,n\}}$ and $\{\tilde{F}_i(\boldsymbol{\phi})\}_{\{i=1,\dots,n\}}$ are vectors in some finite dimensional space. Furthermore the $\tilde{F}_i$ are polynomials in ϕ_j and ε is a parameter which measures the intensity of the noise. Quite generally, $\tilde{\boldsymbol{F}}$ can be decomposed in the following way

$$\tilde{\boldsymbol{F}} = -\frac{\partial V}{\partial \boldsymbol{\phi}} + \boldsymbol{F}(\boldsymbol{\phi}) \tag{3}$$

where V is an appropriate potential function.

† For $\varepsilon = 0$ this is the type of system considered, for example, in [3].

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Our basic assumption is that there exists a decomposition like (3) in which the potential term provides, for large values of the field $\boldsymbol{\phi}$, a restoring force which, in a sense to be made more precise later, will dominate the non-gradient part. Nevertheless, this structure permits chaotic behaviour at the deterministic level. As it is, (1) does not make sense due to the singular character of the noise. However, in two dimensions, for polynomial V and $\boldsymbol{F}$, it is sufficient to replace them by Wick ordered expressions to obtain meaningful evolution equations in an appropriate weak probabilistic sense.

Let us add some comments on this point. With respect to the underlying deterministic dynamics, the addition of white noise in equation (1) generates large fluctuations at small scales in space. As soon as ∂V and F are nonlinear, the self-coupling introduced by local powers of the field induces divergent high frequency behaviour which needs to be removed in order to obtain finite solutions. This removal, the renormalization procedure, does not modify the characteristic behaviour of the system and results in a redefinition of the physical parameters describing it. For example, after renormalization, the equations of evolution keep the same form except that only the renormalized ones are meaningful. For fixed physical parameters there is a unique renormalized form of these equations. In general, the renormalized equations can only be described through a limiting procedure. However, in two spacetime dimensions, for polynomial $\tilde{\boldsymbol{F}}$, the renormalized expression can be analytically written and just results in replacing powers of the field by their Wick ordered equivalent.

2. A simple model

To make things concrete, let us take as an example the following system with $\boldsymbol{\phi}$ two-dimensional

$$\partial_t \boldsymbol{\phi}_t = \Delta \boldsymbol{\phi}_t + A(\boldsymbol{\phi}_t) \cdot \boldsymbol{\phi}_t + \varepsilon \partial_t \boldsymbol{W}_t \tag{4}$$

where A is the 2×2 matrix

$$\begin{pmatrix} \lambda_1(1-|\boldsymbol{\phi}|^2) & -\lambda_2(1-|\boldsymbol{\phi}|^2) \\ \lambda_2(1-|\boldsymbol{\phi}|^2) & \lambda_1(1-|\boldsymbol{\phi}|^2) \end{pmatrix} \tag{5}$$

with $\lambda_1 > 0$.

Notice that in this case the gradient and non-gradient parts are orthogonal.

It is not difficult to construct examples of reaction-diffusion equations with more complicated behaviour. For example, taking $\boldsymbol{\phi}$ three-dimensional, one can have a structure similar to the Lorenz model.

To simplify the discussion of the model we will introduce as in [1] a regularization in space and carry out the proof of existence of weak solutions for the following modified and renormalized version of the model given by (4) and (5)

$$\begin{aligned} \partial_t \phi_1 &= -(-\Delta+1)^{\rho}\phi_1 + (-\Delta+1)^{-1+\rho}{:}[\lambda_1(1-|\boldsymbol{\phi}|^2)\phi_1 - \lambda_2(1-|\boldsymbol{\phi}|^2)\phi_2]{:} + \varepsilon\partial_t W_1 \\ \partial_t \phi_2 &= -(-\Delta+1)^{\rho}\phi_2 + (-\Delta+1)^{-1+\rho}{:}[\lambda_1(1-|\boldsymbol{\phi}|^2)\phi_2 + \lambda_2(1-|\boldsymbol{\phi}|^2)\phi_1]{:} + \varepsilon\partial_t W_2 \end{aligned} \tag{6}$$

with

$$E(\partial_t W_{t,i}(x)\partial_{t'} W_{t',\kappa}(x')) = \delta_{ik}\delta(t-t')(-\Delta+1)^{-1+\rho}(x-x'). \tag{7}$$

for some $0<\rho<1$ to be chosen later.

The Wick ordering symbol : : means the following:

$$\begin{aligned} &:\boldsymbol{\phi}: = \boldsymbol{\phi} \\ &:|\boldsymbol{\phi}|^2\phi_1: = \phi_1^3 - 3\varepsilon^2 c\phi_1 + \phi_1(\phi_2^2 - \varepsilon^2 c) = :\phi_1^3: + \phi_1{:}\phi_2^2{:} \\ &:|\boldsymbol{\phi}|^2\phi_2: = \phi_2^3 - 3\varepsilon^2 c\phi_2 + \phi_2(\phi_1^2 - \varepsilon^2 c) = :\phi_2^3: + \phi_2{:}\phi_1^2{:} \end{aligned} \tag{8}$$

where $c = C(x,x) = (-\Delta+1)^{-1}(x,x)$. Of course, c is infinite so that the right-hand side of (6) is purely formal.

The original model (4) and (5) corresponds to $\rho=1$. We shall comment on this point later.

Proceeding now as in [1] and [2] we transform (6) into an integral system

$$\boldsymbol{\phi}_t = \boldsymbol{Z}_t + \int_0^t \mathrm{d}s \exp[-(t-s)C^{-\rho}]C^{1-\rho}{:}A(\boldsymbol{\phi}_s)\boldsymbol{\phi}_s{:} \tag{9}$$

where $\boldsymbol{Z}_t$ satisfies

$$\mathrm{d}\boldsymbol{Z}_t = -C^{-\rho}\boldsymbol{Z}_t\,\mathrm{d}t + \varepsilon\partial_t \boldsymbol{W}_t. \tag{10}$$

The next step, which defines the weak dynamics, consists in defining the evolution semigroup by

$$E_{\phi_0}(f(\boldsymbol{\phi}_t)) \overset{\text{def}}{\equiv} E_{\phi_0}(f(\boldsymbol{Z}_t)\exp(\xi_t)) \tag{11}$$

where ϕ_0 is the initial condition and

$$\xi_t = \frac{1}{\varepsilon}\int_0^t ({:}A(\boldsymbol{Z}_s)\boldsymbol{Z}_s{:}, \mathrm{d}\boldsymbol{W}_s) - \frac{1}{2\varepsilon^2}\int_0^t \mathrm{d}s({:}A(\boldsymbol{Z}_s)\boldsymbol{Z}_s{:}, C^{1-\rho}{:}A(\boldsymbol{Z}_s)\boldsymbol{Z}_s{:}) \tag{12}$$

and $f(\boldsymbol{\phi}_t)$ is a functional of $\boldsymbol{\phi}_t(\cdot)$.

In [1] and [2] the control of e^{ξ_t} was obtained by performing first the stochastic integral in (12), which was possible due to the gradient character of the equation.

As we shall see in the next section, under appropriate conditions, the existence of the RHS of (11) can be reduced to the results already established in [1, 2].

3. Existence of weak dynamics

To explain our strategy, let us separate in ξ_t the gradient and the non-gradient parts. This is obtained by decomposing A into its diagonal and non-diagonal parts

$$A = A_{\mathrm{D}} + A_{\mathrm{ND}}. \tag{13}$$

Then taking account of orthogonality

$$\xi_t = \xi_{\mathrm{D},t} + \xi_{\mathrm{ND},t}$$

where

$$\begin{aligned} \xi_{\mathrm{D},t} &= \frac{1}{\varepsilon}\int_0^t ({:}A_{\mathrm{D}}(\boldsymbol{Z}_s)\boldsymbol{Z}_s{:}, \mathrm{d}\boldsymbol{W}_s) - \frac{1}{2\varepsilon^2}\int_0^t \mathrm{d}s\|C^{(1-\rho)/2}{:}A_{\mathrm{D}}(\boldsymbol{Z}_s)\boldsymbol{Z}_s{:}\|^2 \\ \xi_{\mathrm{ND},t} &= \frac{1}{\varepsilon}\int_0^t ({:}A_{\mathrm{ND}}(\boldsymbol{Z}_s)\boldsymbol{Z}_s{:}, \mathrm{d}\boldsymbol{W}_s) - \frac{1}{2\varepsilon^2}\int_0^t \mathrm{d}s\|C^{(1-\rho)/2}{:}A_{\mathrm{ND}}(\boldsymbol{Z}_s)\boldsymbol{Z}_s{:}\|^2. \end{aligned} \tag{14}$$

Following [1], to prove the existence of the evolution we need to show that

$$E_{\phi_0}(\mathrm{e}^{p\xi_t})<\infty \tag{15}$$

for some $p>1$.

Our approach is inspired by the discussion of the Girsanov formula in chapter 7 of [4].

We first regularize ξ_t by substituting Z_t with $Z_t^{(N)}$, its projection to the first N eigenvectors of an orthogonal basis, and $C(x, y)$ with $C_N(x, y)$ and define in general

$$C_N^\alpha(x,y) \overset{\mathrm{def}}{=} \sum_{k=1}^{k=N} \lambda_k^{-\alpha}\Phi_k(x)\Phi_k(y). \tag{16}$$

For example, we can take for λ_k and Φ_k the system of eigenvalues and of eigenvectors of $(-\Delta+1)$ in the volume Λ. We have now

$$E_{\phi_0}(\exp(p\xi_t^{(N)}))=E_{\phi_0}(\exp(K_1^{(N)}+K_2^{(N)})) \tag{17}$$

where

$$K_1^{(N)}=\frac{p}{\varepsilon}\int_0^t(:A_{\mathrm{ND}}(Z_s^{(N)})Z_s^{(N)}:,\mathrm{d}W_s)-\frac{p^3}{2\varepsilon^2}\int_0^t \mathrm{d}s\|C_N^{(1-\rho)/2}:A_{\mathrm{ND}}(Z_s^{(N)})Z_s^{(N)}:\|^2$$

and

$$K_2^{(N)}=\frac{p}{\varepsilon}\int_0^t(:A_{\mathrm{D}}(Z_s^{(N)})Z_s^{(N)}:,\mathrm{d}W_s)-\frac{p}{2\varepsilon^2}\int_0^t \mathrm{d}s\|C_N^{(1+\rho)/2}:A_{\mathrm{D}}(Z_s^{(N)})Z_s^{(N)}:|^2$$
$$+\frac{p(p+1)(p-1)}{2\varepsilon^2}\int_0^t \mathrm{d}s\|C_N^{(1+\rho)/2}:A_{\mathrm{ND}}(Z_s^{(N)})Z_s^{(N)}:\|^2.$$

Applying the Hölder inequality

$$E_{\phi_0}(\exp(p\xi_t))\leqslant[E_{\phi_0}(\exp(pK_1^{(N)}))]^{1/p}\left[E_{\phi_0}\left(\exp\left(\frac{p}{p-1}K_2^{(N)}\right)\right)\right]^{(p-1)/p}. \tag{18}$$

The important observation which justifies the splitting of ξ_t is that

$$E_{\phi_0}(\exp(pK_1^{(N)}))=1. \tag{19}$$

This follows from:

(i) $pK_1^{(N)}$ is the Girsanov exponent corresponding to the drift

$$p^2C_N^{(1-\rho)/2}:A_{\mathrm{ND}}(Z^{(N)})Z^{(N)}:$$

(ii) the corresponding stochastic (finite-dimensional) equation has a strong solution for any finite time.

We want to show now that under suitable conditions the second factor in (18) is uniformly bounded in N.

In fact

$$E_{\phi_0}\left(\exp\left(\frac{p}{p-1}K_2^{(N)}\right)\right)\leqslant E_{\phi_0}\left(\exp\left(\frac{p^2}{\varepsilon(p-1)}\int_0^t(:A_{\mathrm{D}}(Z_s^{(N)})Z_s^{(N)}:,\mathrm{d}W_s)\right)\right)\mathrm{e}^M \tag{20}$$

provided

$$-\int_0^t \mathrm{d}s\|C_N^{(1-\rho)/2}:A_{\mathrm{D}}(Z_s^{(N)})Z_s^{(N)}:\|^2$$
$$+(p^2-1)\int_0^t \mathrm{d}s\|C^{(1-\rho)N/2}:A_{\mathrm{ND}}(Z_s^{(N)})Z_s^{(N)}:\|^2<M. \tag{21}$$

This condition is clearly satisfied in our model if $p-1$ is sufficiently small (in fact, if $\lambda_2^2(p^2-1)<\lambda_1^2$). The RHS of (20) is bounded in N as it follows from explicit calculation of the stochastic integral and application of the estimates of [1] valid for $\rho<\frac{1}{10}$. As in [1] this result holds μ_{C_ε} a.e. in the initial condition ϕ_0, where μ_{C_ε} is the Gaussian measure with covariance $\varepsilon^2(-\Delta+1)^{-1}$.

4. Generalizations

The result of the previous section is not limited to the particular model (6). We want to describe here the class of equations for which the extension is straightforward. Consider the system

$$\partial_t\boldsymbol{\phi}=-(-\Delta+1)^{\rho}\boldsymbol{\phi}+(-\Delta+1)^{-1+\rho}:-\frac{\partial V}{\partial\boldsymbol{\phi}}+\boldsymbol{F}:+\varepsilon\partial_t\boldsymbol{W}. \tag{22}$$

$\boldsymbol{\phi}$ is an n-dimensional vector, $\partial_t\boldsymbol{W}$ satisfies (7) and $V(\boldsymbol{\phi})$ and $F(\boldsymbol{\phi})$ are local polynomials in ϕ_i. V is assumed to be bounded below and increasing for $|\boldsymbol{\phi}|\to\infty$ like some even polynomial.

If $(\partial V/\partial\boldsymbol{\phi}, C^{1-\rho}\boldsymbol{F})=0$ and

$$-\left\|\frac{\partial V^{(N)}}{\partial\boldsymbol{\phi}_s}C_N^{(1-\rho)/2}\right\|^2+(p^2-1)\|\boldsymbol{F}^{(N)}(\boldsymbol{\phi}_s)C_N^{(1-\rho)/2}\|^2<M \tag{23}$$

where M is a constant independent of N, our previous analysis applies almost verbatim†.

If $(\partial V/\partial\boldsymbol{\phi}, C^{1-\rho}\boldsymbol{F})\neq 0$, (23) must be replaced by

$$-\left\|\frac{\partial V^{(N)}}{\partial\boldsymbol{\phi}_s}C_N^{(1-\rho)/2}\right\|^2+(p^2-1)\left[\|\boldsymbol{F}^{(N)}(\boldsymbol{\phi}_s)C_N^{(1-\rho)/2}\|^2+2\left(\boldsymbol{F}^{(N)}, C_N^{1-\rho}\frac{\partial V^N}{\partial\boldsymbol{\phi}}\right)\right]<M. \tag{24}$$

We notice that if we call $pK_1^{(N)}$ the Girsanov exponent associated to the drift $p^2C_N^{(1-\rho)/2}:F^{(N)}:$, the condition

$$E\left(\int_0^t\|p^2C_N^{(1-\rho)/2}:F^{(N)}\|^2\,\mathrm{d}s\right)<\infty$$

which is certainly satisfied for polynomial $F^{(N)}$, implies [4]

$$E(\exp(pK_1^{(N)}))\leqslant 1. \tag{25}$$

In conclusion the following theorem has been proved.

Theorem 1. For $\rho<\frac{1}{10}$, the system (22) with the above hypotheses on V and $\boldsymbol{F}$, under the condition (24), possesses a weak dynamics defined by

$$E_{\phi_0}(f(\boldsymbol{\phi}_t))\stackrel{\text{def}}{\equiv}E_{\phi_0}(f(\boldsymbol{Z}_t)\,\mathrm{e}^{\xi_t}) \tag{26}$$

where $f(\boldsymbol{Z}_t)$ is a functional of $\boldsymbol{Z}_t$ belonging to $L^q(\mathrm{d}\mu_{ou})$ for an appropriate q, and $\mathrm{d}\mu_{ou}$ is the Ornstein-Uhlenbeck measure corresponding to the solution of (10).

† Actually in (23) and (24) we can allow a growth of M with N provided it is not too strong.

Everywhere in our equations we have included the parameter ε representing the intensity of the noise. In many physical situations, one is interested in the limit when ε is small. In the case of the ordinary stochastic differential equations there is a well developed theory of small stochastic perturbations which shows that important informations on the deterministic part of the equations are obtained by studying the limit when $\varepsilon \to 0$ and that very interesting phenomena take place for ε small. In [2] it was shown that the theory of small random perturbations can be adapted, up to a certain extent, to the case of gradient systems in two space dimensions. The results of the present paper imply that the extension holds also for non-gradient systems with the same limitations. These limitations come from the fact that due to the ultraviolet divergencies the random perturbation is always strong at sufficiently small space scale.

Let us conclude with some comments. The techniques of [1, 2] do not apply when $\rho = 1$ due to the circumstance that the measure corresponding to the nonlinear equations is not absolutely continuous with respect to that generated by the linear part. However, it is reasonable to expect that this difficulty can be overcome by a suitable adaptation of the renormalization group methods developed in constructive field theory. Similarly, one thinks that these methods can be used to handle higher-dimensional cases and the problem of the existence and of the construction of stationary measures.

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